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INVESTIGATING THE DIAGNOSTICS ALGORITHMS OF THE THERMAL EFFECT IN DESIGN

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We examine the iteration solution algorithms of inverse heat-conduction problems (IHCP) with consideration of some a priori information regarding the sought relationship.

In recent times, the derivation of the characteristics applicable to heat-exchange processes based on the methodology of inverse problems has gained increasing applicability, both in the processing of experimental results, as well as in the construction of mathematical models of real processes. This has stimulated further development of solution algorithms for inverse heat-exchange problems and more extensive investigation of their properties from the standpoint of practical application.

Let us formulate some inverse problems in the form

$$Au = f,\ u \in U,\ f \in F, \tag{1}$$

where A: U → F is a nonlinear operator in the general sense; U and F are Hilbert spaces. We know from physical considerations that u, as a rule, is a smooth function. Therefore, for U we make use of the Sobolev space W_2^k. The function f, since it is a result of measurements, is generally known with some error and represents a rather arbitrary relationship f_δ. Naturally in this case the space L_2 must be examined from the standpoint of F.

The operator that is the reciprocal of A is usually bounded, i.e., the formulated problem is incorrect and for its solution we must make use of regularizing algorithms. In the following, for this purpose, we employ a method based on iteration regularization. Research has shown [1] that excellent effectiveness is achieved by IHCP solution algorithms based on a scheme from the method of conjugate gradients, where the iteration number k is taken as the regularization parameter

$$u_{k+1} = u_k - \beta_k p_k,\ k = 0,\ 1,\ \ldots,\ K^*, \tag{2}$$

where the direction of descent

$$p_{k+1} = J'_{W_2^k}(u_{k+1}) + \gamma_{k+1} p_k, \tag{3}$$

$$\gamma_0 = 0;\ \gamma_{k+1} = -\frac{(J'_{W_2^k}(u_{k+1}),\ J'_{W_2^k}(u_k) - J'_{W_2^k}(u_{k+1}))_{W_2^k}}{\| J'_{W_2^k}(u_k) \|^2_{W_2^k}};$$

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K* is the number of the following iteration, which is chosen according to some law which provides for the regularization of the method (for example, the disparity rule [1]); $J'_{W_2}k(u)$ is the gradient of the functional $J(u) = \|Au - f_\delta\|_{L_2}^2$ in the space W_2^k. The descent interval in this case is determined from the condition

$$\beta_k = \operatorname{argmin} J(u_k - \beta p_k), \ \beta > 0. \tag{4}$$

At the present time the IHCP solution algorithms have found widespread application, and here the iteration sequence (2) is constructed in the space of the functions L_2 without making use of available information regarding the unique features of the reproducibility of u. A more exact solution for the incorrect IHCP can be attained by means of algorithms which make possible consideration of similar a priori information, in particular, data regarding the smoothness of the sought function, as well as its quantitative characteristics and individual points within the determination region. A number of papers [1-7] has been devoted to the development of such algorithms.

In the following we examine a means of providing for the a priori information offered in [8, 9]. The iteration sequence (2) is constructed in the space of the functions W_2^k. In this case, the gradient of the functional $J'_{W_2}k$ in this space is determined on the spaces of known values of the gradient J'_{L_2}, corresponding to the space L_2 by means of solution of the boundary-value problem

$$\sum_{n=0}^{k}(-1)^n \frac{d^n}{dx^n}\left(r_n \frac{d^n J'_{W_2^k}}{dx^n}\right) = J'_{L_2}(x), \ x \in (0, b). \tag{5}$$

When the sought relationship is found among the functions belonging to the space W_2^1, differential equation (5) will be most simple:

$$r_0 J'_{W_2^1}(x) - (r_1 (J'_{W_2^1}(x))_x)_x = J'_{L_2}(x), \ x \in (0, b). \tag{6}$$

The boundary conditions depend on the type of information regarding the sought function that is given. It is natural that the initial approximation $u_0(x)$ must be a function of corresponding smoothness and must agree with the boundary conditions chosen.

We examine three widely encountered cases of given values for the derivatives or for the actual functions themselves at the boundaries 0 and b. The solution to Eq. (5), with the corresponding boundary conditions, can be achieved analytically by means of the method of the Green's function.

If we use the information regarding the values of the derivatives of the sought function $u_x(0) = a_1$ and $u_x(b) = a_2$, we must assume the following boundary conditions

$$(J'_{W_2^1}(x))_x|_{x=0, b} = 0 \tag{7}$$

and take the function $u_0(x)$ as the initial approximation, containing the corresponding derivatives at the boundaries of the region. The corresponding solution to problem (6)-(7) is obtained in the form

$$J'_{W_2^1}(x) = c \operatorname{ch}(Rx) - \frac{1}{Rr_1}\int_0^x J'_{L_2}(\xi) \operatorname{sh}[R(x-\xi)]\, x\xi, \tag{8}$$

where

$$c = (2Rr_1 \operatorname{sh}(Rb))^{-1}\int_0^b J'_{L_2}(\xi) \operatorname{ch}[R(b-\xi)]\, d\xi; \tag{9}$$

$$R = \sqrt{r_0/r_1}.$$

When $u_x(x)|_{x=0} = a_1$, $u(b) = q_2$ is given, the initial approximation $u_0(x)$ must be characterized by these same parameters at the points x = 0, b, and the boundary conditions become as follows:

$$(J'_{W_2^1})_x|_{x=0}=0;\ J'_{W_2^1}(x)|_{x=b}=0. \tag{10}$$

In this case, the solution to problem (5)-(10) has the form of (8) with the constant

$$c=(2Rr_1\operatorname{ch}(Rb))^{-1}\int_0^b J'_{L_2}(\xi)\operatorname{sh}[R(b-\xi)]\,d\xi. \tag{11}$$

Finally, if we have information regarding the values of the functions at the boundaries of the region $u(0) = q_1$ and $u(b) = q_2$, then we can use the linear relationship $u_0(x)$ as the initial approximation and this linear function is characterized by these same parameters. The boundary conditions for Eq. (6) assume the form

$$J'_{W_2^k}(x)|_{x=0,\,b}=0, \tag{12}$$

while the solution is given by the formula

$$J'_{W_2^1}(x)=c\operatorname{sh}(Rx)-\frac{1}{Rr_1}\int_0^x J'_{L_2}(\xi)\operatorname{sh}[R(x-\xi)]\,d\xi, \tag{13}$$

where

$$c=(2Rr_1\operatorname{sh}(Rb))^{-1}\int_0^b J'_{L_2}(\xi)\operatorname{sh}[R(b-\xi)]\,d\xi. \tag{14}$$

With a solution in space $W_2{}^2$ we have a fourth-order differential equation

$$r_0J'_{W_2^2}-(r_1(J'_{W_2^2})_x)_x+(r_2(J'_{W_2^2})_{xx})_{xx}=J'_{L_2}(x) \tag{15}$$

with boundary conditions

$$r_1(J'_{W_2^2})_x-(r_1(J'_{W_2^2})_{xx})_x|_{x=0,\,b}=0; \tag{16}$$

$$r_2(J'_{W_2^2})_{xx}|_{x=0,\,b}=0. \tag{17}$$

The solution of problem (15)-(17), together with the method of the Green's function, has real values when the coefficients r_0, r_1, and r_2 satisfy the condition

$$D_1=r_1^2-4r_0r_2>0, \tag{18}$$

and is written in the form

$$J'_{W_2^2}(x)=c_1e^{Ax}+c_2e^{-Ax}+c_3e^{Bx}+c_4e^{-Bx}+S(x), \tag{19}$$

where

$$\begin{gathered}A=\sqrt{D_2^+/2r_2}\ ;\ B=\sqrt{D_2^-/2r_2}\ ;\ D^{\pm}=r_1\pm\sqrt{D_1};\\ S(x)=[r_2(B^2-A^2)]^{-1}\int_0^x J'_{L_2}(\xi)[B^{-1}\operatorname{sh}B(x-\xi)-A^{-1}\operatorname{sh}A(x-\xi)]\,d\xi.\end{gathered} \tag{20}$$

The coefficients contained in expression (19) are calculated through solution of the following system of equations:

$$\begin{gathered}c_1A(1-A^2)-c_2A(1-A^2)+c_3B(1-B^2)-c_4B(1-B^2)=0;\\ c_1A(1-A^2)e^{bA}-c_2A(1-A^2)e^{-bA}+c_3B(1-B^2)e^{bB}-c_4B(1-\\ -B^2)e^{-bB}=-P;\\ c_1A^2+c_2A^2+c_3B^2+c_4B^2=0;\\ c_1A^2e^{bA}+c_2A^2e^{-bA}+c_3B^2e^{bB}+c_4B^2e^{-bB}=-Q,\end{gathered} \tag{21}$$

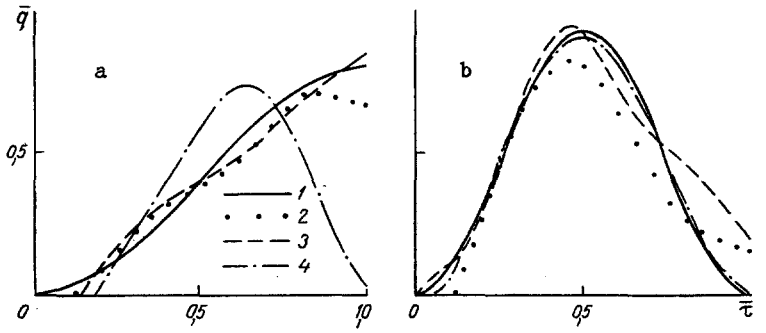

Fig. 1. Determination of the functions $\bar{q}(\bar{\tau})$ (1) for the original data perturbed in accordance with the normal law ($3\sigma = 0.1\ T_{max}$) by means of the solution algorithms in the spaces W_2^1 (2), W_2^2 (3), L_2 (4) for $0.012 \leq \Delta Fo \leq 0.038$.

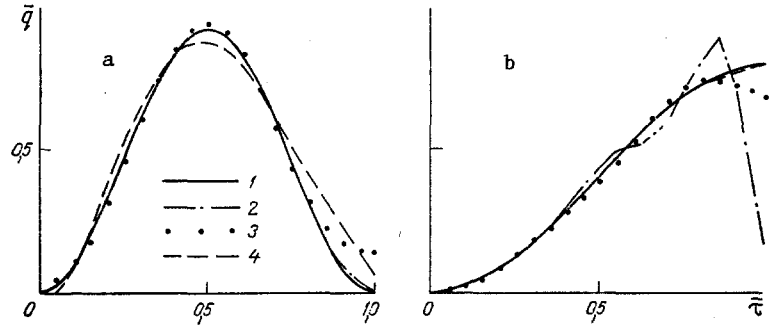

Fig. 2. Determination of the functions $\bar{q}(\bar{\tau})$ (1) on the unperturbed initial data by means of the solution algorithms in the spaces L_2 (2), W_2^1 (3), W_2^2 (4) for $0.012 \leq \Delta Fo \leq 0.038$.

where

$$P = [r_2(B^2 - A^2)]^{-1} \int_0^b J'_{L_2}(\xi)\,[(1 - B^2)\operatorname{ch} B(b - \xi) - (1 - A^2)\operatorname{ch} A(b - \xi)]\,d\xi; \tag{22}$$

$$Q = [r_2(B^2 - A^2)]^{-1} \int_0^b J'_{L_2}(\xi)\,[B \operatorname{sh} B(b - \xi) - A \operatorname{sh} A(b - \xi)]\,d\xi. \tag{23}$$

It should be noted that the cited formulas for the calculation of the gradients of the functional in spaces W_2^1 and W_2^2 on the basis of the gradient found in space L_2 can be utilized to solve the various inverse problems. Later on we will deal with the solution of a specific problem, and namely the nonlinear boundary IHCP of determining the density of the heat flux as a function of time $u(x) = q(\tau)$ at one of the surfaces of an unbounded plate on the basis of the data at one or more points, with consideration of the above-considered a priori information. The gradient of the functional in the space L_2 in this case is determined by the method presented in [2].

The investigation carried out by means of calculation experiments for unperturbed and perturbed initial data and various thermophysical characteristics of materials enabled us to carry out a comparison of the results obtained in the spaces W_2^2, W_2^1, and L_2 in determining the functions $q(\tau)$ of various forms, as well as to ascertain the conditions for the selection of the approximate solution in the case in which the magnitude of the error in the initial information is unknown.

The derived results indicated that the solution algorithms in spaces W_2^2 and W_2^1, with consideration given to the information regarding smoothness, exhibit smoothing properties and are more stable with regard to errors in temperature. Unlike the solution algorithm in space L_2, which is characterized by lower viscosity, as the number of iterations in these

increases there is no development of oscillations, but rather only some "waviness" occurs, a consequence of the measurement errors (Fig. 1), i.e., they exhibit a rather markedly expressed property of self-regularization. In practical applications it proved to be possible to use the proximity condition of the iteration approximations to reduce the number of calculations:

$$|\sqrt{J_k}-\sqrt{J_{k+1}}|\leqslant\varepsilon; \tag{24}$$

$$|\sqrt{J_k}-\sqrt{J_{k+2}}|\leqslant\varepsilon. \tag{25}$$

With the solution in space L_2, condition (24) makes it possible to achieve excellent approximation for the case in which the parameter $\Delta Fo \geq 1$. This parameter takes into consideration the thermal depth to which the thermocouples are positioned, as well as the time-discretization interval $\Delta\tau$. In the nonlinear case, it is calculated in accordance with the formula

$$\Delta Fo=\min_T[\lambda(T)\Delta\tau/(c(T)d^2)]. \tag{26}$$

When $\Delta Fo < 1$, the selection of the approximate solution in space L_2 can be found on the basis of the proximity condition for the functional with respect to the value of the errors in the initial temperature data [1] (Fig. 1b)

$$\sqrt{J_K}\approx\delta_T \tag{27}$$

or, if the error δ_T is unknown, it can be found through additional measurements or on the basis of the increment of the functional [10].

If the initial approximation $q_0(\tau)$ is chosen arbitrarily, then in the solution of the IHCP in space L_2 it is not always possible to obtain a reliable result, even with exact initial data (Fig. 2b). At the same time, the algorithms that we considered above and which make use of information regarding the smoothness enable us to obtain a rather close approximation of the sought function.

As an example, Fig. 2 shows the results from the determinations of two different functions for the case of unperturbed initial data. In both cases, we have taken the value of $q_0 = 0$ as the initial data. The cited results show that together with the solution algorithm in space L_2 it is the function $q(\tau)$, shown in Fig. 2a, that is determined more accurately, whereas in the determination of the second function (Fig. 2b) the result differs significantly from the exact value.

The solution algorithms in spaces W_2^1 and W_2^2, with consideration of the smoothness of the sought results, based on the boundary-value problems (6)-(7) and (15)-(17), yielded satisfactory approximate relationships both for the exact and the perturbed initial data for both of the functions $q(\tau)$ under examination. And here, the more exact solutions are found in the function space W_2^2 (see Figs. 1 and 2).

NOTATION

$u = u(x)$, sought function; $(\cdot)_x$, first derivative with respect to the argument; $(\cdot)_{xx}$, second derivative with respect to the argument; u_x, derivative with respect to x; ε, positive small number; $\lambda(T)$ and $c(T)$, specific heat capacity and volumetric heat capacity of the material; d, coordinate for the sensor installation, whose measurements are used in the solution of the IHCP; δ_T, error in temperature measurements.

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A UNIVERSAL ALGORITHM FOR THE SOLUTION OF PROBLEMS INVOLVING THE MATHEMATICAL MODELING OF THE THERMAL REGIME IN A STRUCTURE, IN ONE-DIMENSIONAL APPROXIMATION

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We examine a universal solution algorithm for problems related to the mathematical modeling of the heat regime in structures in one-dimensional approximation, synthesizing the possibilities and advantages of the solution algorithms of these problems, as determined from graphs of general form and a graph in the form of a tree.

The method of mathematical modeling of the thermal regime in structures in one-dimensional approximation [1-3] has recently found widespread application with regard to problems of thermal designs in various heat-engineering systems and devices. The thermal model of a structure in this case is represented in the form of a graph, on N of whose arms are given the equations of heat conduction modeling the thermal state in distributed structural elements, with the heat-balance equation for concentrated elements in combination with conditions of thermal stress given at the N_V apices (at $N_{\alpha_{in}}$ internal apices of the graph) or by the boundary conditions (at N_{α_b} boundary apices). We will identify the boundary apices of the graph as those apices with which only a single arm is associated. The inside apices will include all those with which a minimum of two arms are connected.

As a rule, the system of nonsteady nonuniform one-dimensional heat-conduction equations with which we are dealing here, as a result of the finite-difference approximation of the differential operators, reduces to a system of algebraic equations determined on the graph of the thermal model for the solution of which various modifications of the parametric sweeping method [2] is used, or where use is made of a generalized algorithm [3], utilizing a cyclical sweeping method. These methods exhibit excellent convergence and stability and are suitable for thermal models whose graphs are arbitrary in form (see Fig. 1a), i.e., it contains cycles, loops, etc. The application of these methods requires a considerable number of arithmetic operations and, consequently, considerable computer capacity.

The original graph can frequently be represented as some combination (total) of simpler interconnected graphs. This makes it possible to break down the graph in the following manner. Let us assume that N_b boundary graphs are contained within the original graph; these boundary graphs simulate the characteristics of a tree (or bush), and there is also a root graph of general form (i.e., with loops and cycles) to link all of the separate graphs into a graph representing the thermal model of the structure. In this case, the root apices of the boundary graphs are the inside apices of the root graph. If we were to include the simplest tree-shaped graphs consisting of a single arm and two terminal apices in the number of boundary graphs, all of the apices of the root graph would be inside graphs in terms of the earlier-introduced definition.

The α apices of the graph under consideration have been determined on the set $V = V_r + \sum_{p=1}^{N_r} V_p$. The total number of N_V apices in the original graph consists of the N_{α_r} apices

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